Swansea University
Mathematics scholarship exam 2018

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $3\cdot(67)^\bullet$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 4/11 as a repeating decimal expansion.

2. Determine whether 673 is prime or not. Show your reasoning.

3. Solve the following simultaneous linear equations for x and y:

$$x + 5y = 4\ ,\ \ 2x + 3y = -6\ .$$

4. The number of bacteria in a culture increases by 6% every minute at a constant rate. If there were initially two million bacteria in the culture, how many would there be after 220 seconds?

5. Evaluate the integral

$$\int_0^1 \sin(x+1)\ \mathrm{d}x\ .$$

6. There is one real number x which satisfies $x = 1 - x^4$ and $0 \le x \le 1$. Calculate x to within an error of $\frac{1}{16}$ by any method.

7. Find the differential with respect to x of

$$x^3 + 2\,x^2 + x\ .$$

8. Find the differential with respect to x of

$$\sin(x^2)\ .$$

9. What is the area of an isosceles triangle with sides 6, 8 and 8?

10. Given that $x = 2$ is a root of the cubic $x^3 + x^2 - 5\,x - 2$, find the other two roots.

Section B

1. The Fibonacci numbers F_n are defined by an infinite sequence beginning

$$F_0 = 0\ ,\ F_1 = 1\ ,\ F_2 = 1\ ,\ F_3 = 2\ ,\ F_4 = 3\ ,\ F_5 = 5\ ,\ F_6 = 8\ ,\ F_7 = 13\ ,\ F_8 = 21\ ,$$

where the rule is to add the two preceding numbers to get the next one, i.e. $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

a) Show that the following formula (*) gives the correct values for F_0 and F_1:

$$(*) \qquad F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

b) Show that if the formula (*) gives the correct values for F_{n-1} and F_{n-2} then it also gives the correct value of F_n.

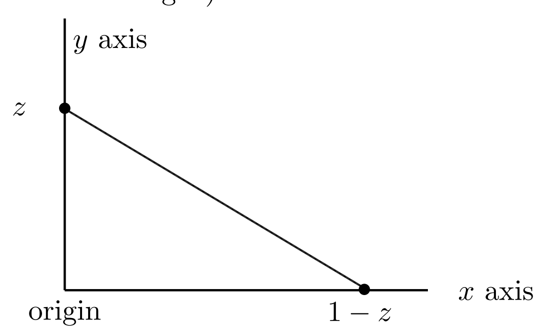
c) The numbers a_n are defined by $a_1 = 1$ and then the formula for $n \geq 1$

$$a_{n+1} = \frac{1}{1 + a_n}\ .$$

Write down the values of a_2, a_3 and a_4.

d) Find a formula for a_n in part (c) in terms of a ratio of Fibonacci numbers. (You need not prove that the formula is correct.)

2. The following picture shows a triangle formed by part of the x axis, part of the y axis and the line from $(0, z)$ to $(1 - z, 0)$ for some value $z \in [0, 1]$. (The origin is the third vertex of the triangle.)



Answer the following questions, remembering to show your working:

a) What is the area of the triangle as a function of z?
b) Find the value of $z \in [0, 1]$ which maximises the area of the triangle.
c) What is the perimeter of the triangle as a function of z?
d) Find the value of $z \in [0, 1]$ which minimises the perimeter of the triangle.
e) What is the minimum value of the perimeter?

3. A particle of mass m rests on a rough plane which is inclined at an angle θ to the horizontal. The coefficient of friction between the plane and the particle is μ. A horizontal force of magnitude P is applied to the particle and is just sufficient to prevent the particle from sliding down the slope. If however a force of magnitude $2P$ is applied parallel to the slope and acting up the slope, then the particle is just on the point of slipping up the slope. If instead a particle of mass $2m$ sits on the slope then a force $2P$ up the slope is just sufficient to prevent the particle from sliding down. Find the values of θ, μ and P in terms of m and g where g is the acceleration due to gravity.

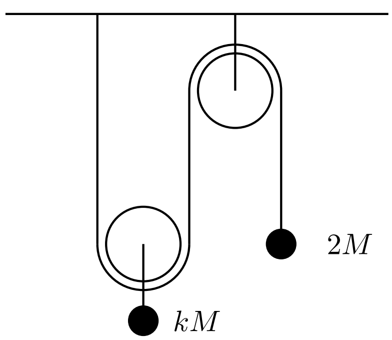
4. Find the following infinite sums, where x is a real number in the interval $-1 < x < 1$.

$$a) \quad \sum_{n=1}^{\infty} \frac{x^n}{n} \,, \qquad b) \quad \sum_{n=1}^{\infty} n\, x^n \,.$$

You may assume the following formula, but must explain your answers from there:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \,.$$

5. One end of a light inextensible string is attached to the ceiling. The string passes under a smooth light pulley which is carrying a mass kM. The pulley is free to move and is suspended only by the string. The string then passes over a fixed light smooth pulley and at the other end a particle of mass $2M$ is attached. The system is released from rest. Find the tension in the string and the acceleration of the movable pulley in terms of k, M and g where g is the acceleration due to gravity. Show that the particle of mass $2M$ will ascend if $k > 4$.



6. A logician visits the island of Knights and Knaves. Remember that Knights always tell the truth, that Knaves always lie, and that every inhabitant is either a Knight or a Knave.

(a) On one occasion the logician meets two inhabitants A and B. He has no previous knowledge of whether either of the inhabitants is a Knight or a Knave. To find out something about this, he asks B the question 'is at least one of you a Knave?'. The reply from B is either 'Yes' or 'No', but because of the noise of a passing train you cannot hear which. However the logician did hear the reply, and remarks 'now I know exactly what A and B are'. Can you work out whether the answer was 'Yes' or 'No'? What are A and B?

(b) On another occasion the logician meets two inhabitants C and D while he is walking to the post office. The meeting takes place at a fork in the road, with two possible ways forward, left or right, and only one of these leads to the post office. The logician asks which way to go to get to the post office.

D says 'C is a Knight'.

C says 'if the way is left, then D is a Knave'.

Which way is the post office? Can you identify C and D?

7. The Gaulish chieftain Combinatorix is going to Rome to see Julius Cæsar, and is taking four druids and six warriors with him.

(a) On the journey to Rome the party walks in single file, with Combinatorix always at the front. Given that the warriors and druids can all be told apart, how many different ways can the party be arranged in a line?

(b) They come to a village with two inns and stay for the night. Combinatorix takes the finest room, leaving the rest of the party to be split between the first inn, which has six places left, and the second inn, which has 4 places left. How many different ways can this splitting between the two inns be done?

(c) The next night they come to another village with two inns and stay for the night. Combinatorix again takes the finest room, leaving the rest of the party to be split between the first inn, which has six places left, and the second inn, which also has six places left. In how many different ways can this splitting between the two inns be done?

(d) The party stays at the village in (c) for a second night. The two innkeepers ask Combinatorix if he can arrange for exactly three warriors to stay in each inn, as they are afraid that too many warriors in one inn could cause trouble. How many different ways can this splitting between the two inns be done, given the restriction on placing the warriors?

(e) On leaving the village, the innkeepers present the four druids in the party with new robes. These robes are identical, and it is now impossible to tell the druids apart. The party again walks in single file, with Combinatorix leading the way, as they leave the village. In how many different distinguishable ways can the party be arranged in a line, given that all the druids now look the same?

8. An experiment is completed in which a golf ball is dropped onto a horizontal surface. The speed of the ball before impact, u, and immediately after impact, v, is measured each time and it is found that regardless of the height from which the ball is dropped the ratio v/u approximately satisfies,

$$\frac{v}{u} \approx \frac{1}{3}.$$

Let v_n denote the speed of the ball immediately after the nth bounce. Deduce that,

$$v_n = \frac{1}{3^{n-1}} v_1.$$

Show that if the ball is dropped and left to bounce 5 times then the time taken from the first bounce to the fifth bounce is,

$$\frac{2v_1}{g\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3}\right)}$$

where v_1 is the speed of the ball immediately after the first bounce.

If the ball was left to continue bouncing until it stopped how long would you have to wait?

9. All dice are six sided, with sides labelled 1,2,3,4,5,6 as usual, and are assumed to be fair. All throws of the dice are assumed to be independent.

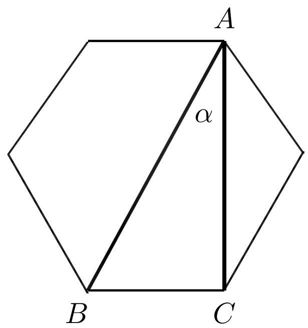
(a) Two six sided dice are thrown, and the numbers on the sides are added. What is the probability that the sum is 7? What is the probability that the sum is 6?

(b) Two six sided dice are thrown by another person, who tells you that the sum of the dice is 6 (you cannot see the dice). Given that information, what is the probability that both dice were 3?

(c) Two six sided dice are thrown. If the numbers on both dice are the same, and only in this case, a third dice is thrown. What is the probability that the sum of all the dice is 7? What is the probability that the sum is 6?

(d) The game in (c) is played by another person, and you cannot see the dice. That person says that the total on all the dice thrown was 7. Given that information, what is the probability that three dice were thrown?

10. The following figure is a regular hexagon, with side (e.g. BC) length 1.



Answer the following questions about the labelled vertices A, B, C. Explain your method.

a) What is the length of AC?

b) What is the length of AB?

c) What is the angle α of the triangle BAC?

END OF EXAM