Swansea University
Mathematics scholarship exam 2019

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $5 \cdot (04)^\bullet$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 6/7 as a repeating decimal expansion.

2. Determine whether 667 is prime or not. Show your reasoning.

3. Solve the following simultaneous linear equations for x and y:

$$2x + y = 7\ ,\ 5x + 2y = 8\ .$$

4. The number of bacteria in a culture increases by 5% per minute at a constant rate. If there were initially two million bacteria in the culture, how many would there be after 200 seconds?

5. Evaluate the integral

$$\int_0^\pi (\cos(x) + 1)\ \mathrm{d}x\ .$$

6. There is one real number x which satisfies $\cos(x) = x$ and $0 \le x \le 1$ (use radians for the cosine). Calculate x to within an error of $\frac{1}{16}$ by any method.

7. Find the differential with respect to x of

$$e^{2x-2} + x\ .$$

8. Find the differential with respect to x of

$$\tan(x^3 + x)\ .$$

9. A triangle has sides of length $a = 5$, $b = 6$ and $c = 4$. Find the angle opposite side b.

10. Given that $x = 3$ is a root of the cubic $x^3 - 12\,x + 9$, find the other two roots.

Section B

1. A sequence of real numbers x_n for integer $n \geq 1$ is given by

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{2\,x_n} \,.$$

a) If $x_1 = 2$, find x_2 and x_3.

b) If $x_n = x_{n+1}$ then x_n can only take two values. Find these values.

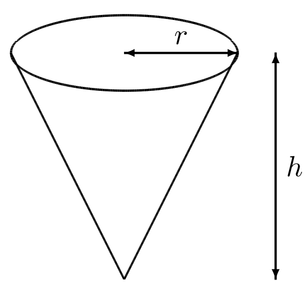
c) Suppose that x_n is any real value with $x_n > 1$. Show that $x_{n+1} < x_n$.

d) Show that

$$x_{n+1} - 1 = \frac{(x_n - 1)^2}{2\,x_n} \,.$$

e) From (d) deduce that if $1 < x_n < 1 + \delta$, then $1 < x_{n+1} < 1 + \delta^2/2$.

2. An ice cream cone manufacturer can make cones of any height h and any radius r of the circle as shown. The area of the wafer is $\pi\, r\, \sqrt{r^2 + h^2}$ and the volume of the cone is $\frac{1}{3}\,\pi\, r^2\, h$. (For packing reasons the ice cream is filled in level with the top of the cone, and there is only wafer on the curved surface, not on top.)



a) For a particular product the amount of wafer that can be used is limited, so the area of the wafer (i.e. the area of the curved surface of the cone) is fixed at a specified value a. Rearrange the formula for the area above so that h is given as a function of r and a.

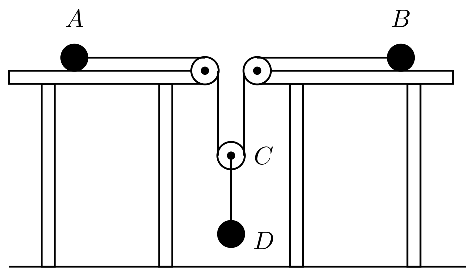
b) Use this value of h to give a function for the volume in terms of r and a.

c) Differentiate the volume (specified in terms of r and a) with respect to r, given that a is a constant.

d) Find the value of $r > 0$ which maximises the volume for a given value of $a > 0$. (You may assume that the zero of the derivative of the volume with respect to r really is a maximum, without further proof.)

3. Two particles A and B of mass $5Kg$ and $3Kg$ respectively rest on two horizontal rough tables as shown below. Each particle is attached to one end of a light inextensible string which passes under a smooth light pulley C which carries a particle D of mass $4Kg$.

Suppose that the coefficient of friction μ is the same for both A and B. If μ is large enough to just stop A from moving but is not large enough to stop B from sliding how big is μ? How large would μ have to be to stop the whole system from moving?



4.

a) State the principle of induction (any version will do).

b) The sequence of integers a_n for $n \geq 1$ is given by $a_1 = 1$ and the recursive equation $a_{n+1} = a_n + 2\,n + 2$ for $n \geq 1$. Write down the values of a_2 and a_3.

c) Find values of constants A and B so that $a_n = n^2 + A\,n + B$ fits the first few values of a_n.

d) Show by induction that the formula for a_n, with your determined values of A and B, is a solution of the recursive equation in (a) for all integer $n \geq 1$.

5. There are two cannons located on a large flat field. The cannons are separated by 10m and are pointed towards each other. Each cannon is pointed at an elevation of α radians to the horizontal. The first cannon is fired and then 1 second later the second cannon is fired. Given that both cannons fire balls at a speed of u m/s, show that the balls will collide in mid air if,

$$u = \sqrt{\frac{10g}{\sin(2\alpha)}},$$

and α is such that,

$$2u \sin\alpha > g.$$

where g is the acceleration due to gravity.

6. Inspector Code of the Swansea police is investigating a crime on the island of Knights and Knaves. Remember that Knights always tell the truth, that Knaves always lie, and that all the inhabitants of the island are either Knights or Knaves.

(a) The inspector meets an inhabitant who says 'If I am a Knight, then I am guilty'. Is the inhabitant guilty, or is there insufficient information to tell?

(b) The inspector meets another inhabitant who says 'If I am a Knave, then I am guilty'. Is the inhabitant guilty, or is there insufficient information to tell?

(c) The inspector meets two more inhabitants, P and Q, who make statements:

P says: If I am guilty, then both of us are guilty.
Q says: Exactly one of us is innocent.
P says: Q is innocent.

State whether Q is innocent or guilty, and also whether Q is a Knight or a Knave.

(d) The inspector meets another inhabitant who says 'If I am guilty, then I am a Knight'. Inspector Code replies 'you expect me to say that I cannot tell if you are innocent or guilty. However I was given a note by the authorities on the island earlier, telling me whether you were a Knight or a Knave. As a result of that and what you just said, I know definitively whether you are innocent or guilty'. Which is the case, is the inhabitant innocent or guilty?

7. There are 14 pupils in a class.

(a) In how many different ways can the class be arranged in a line?

(b) The class is split into two football teams of seven each, A and B. In how many different ways can the teams be chosen? (The order of any choice is not counted.)

(c) The names of people in the teams A and B are now known. Now a goalkeeper is chosen for team A, and a goalkeeper is chosen for team B. In how many different ways can this be done?

(d) The names of people in the teams A and B, and the goalkeepers, are now known. During a match team A scores two goals (by different players) and team B scores one goal. (The goals were not scored by the goalkeepers, and were not own goals.) After the match a list is made of the goal scorers. How many possible lists could be made? (The order of names is not counted.)

8. Two wooden blocks of thickness $2a$ and $4a$ respectively are held fixed next to each other. A gun fires a pellet of mass m horizontally at the blocks so that the pellet hits the first block perpendicular to its surface. Suppose that as the bullet passes through the first block its motion is resisted by a force F_1 and that as it passes through the second block its motion is resisted by a force F_2.

a) The bullet hits the first block with a speed u and enters the second block with a speed v. It is bought to rest after travelling a distance a into the second block. Find F_1 and F_2 in terms of u, v, m, a.

b) A second bullet of mass m is now fired in the opposite direction so it hits the second block first with a speed u. Show that this bullet will pass through both blocks if $\sqrt{5}v < u$.

9. All dice are six sided, with sides labelled 1,2,3,4,5,6 as usual, and are assumed to be fair. All throws of the dice are assumed to be independent.

(a) One dice is thrown and the number is recorded. What is the probability that this number is 3?

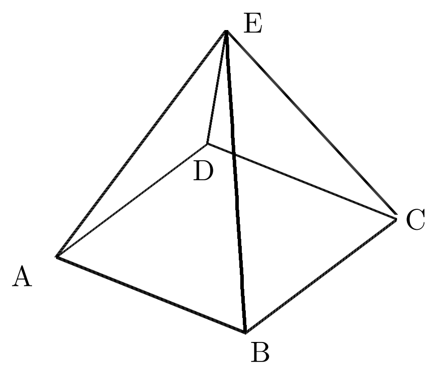
The remaining parts are about the following game: One dice is thrown and the number is recorded. In the case that this number is even, and only in that case, another dice is thrown. The final number is the number on the first dice, if only one dice was thrown, and the sum of the numbers on the two dice if two were thrown.

(b) In one round of this game, what is the probability that this final number is 3?

(c) In another round of this game, suppose that you know that the final number is 3, but you do not know how many dice were thrown. What is the probability that only one dice was thrown?

(d) In another round of this game, suppose that you know that two dice were thrown, but you do not know the final number. What is the probability that the final number is 6?

10. A square based pyramid (with base the horizontal square ABCD in the diagram) has apex (highest point) E, which is directly above the midpoint of the square. The length of AB is 50 metres, and the point E is a height 35 metres above the midpoint of the base square.



(a) Let M be the mid-point of the line AB. What is the length of the line EM?
(b) What is the total surface area of the pyramid (including the base).
(c) What is the length of the line AE?
(d) What is the volume of the pyramid? [If you don't know how to work out the volume of a pyramid, do not spend ages trying to do it!]

END OF EXAM