Swansea University
Mathematics scholarship exam 2020

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $3\cdot(24)^\bullet$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 8/11 as a repeating decimal expansion.

2. Determine whether 629 is prime or not. Show your reasoning.

3. Solve the following simultaneous linear equations for x and y:

$$2x + 3y = 7\ , \quad 3x + 2y = 8\ .$$

4. The number of bacteria in a culture increases by 8% per minute at a constant rate. If there were initially ten million bacteria in the culture, how many would there be after 400 seconds?

5. Evaluate the integral

$$\int_0^\pi \sin(3x)\ \mathrm{d}x\ .$$

6. There is one real number x which satisfies $x^4 + x - 1 = 0$ and $0 \le x \le 1$. Find x to within an error of $\frac{1}{16}$ by any method.

7. Find the differential with respect to x of

$$4\,x^2 + 3 - 5x^{-1}\ .$$

8. Find the differential with respect to x of

$$\sin(x^3)\ .$$

9. A triangle has sides of length $a = 5$, $b = 4$ and $c = 7$. Find the angle opposite side b.

10. Given that $x = 2$ is a root of the cubic x^3+x^2-8x+4, find the other two roots.

Section B

1. The prime numbers $3, 5, 7$ form a triple separated by 2 each time, i.e. are of the form $p, p+2, p+4$. Explain why it is impossible to find another triple of primes of the form $p, p+2, p+4$.

2. A cuboidal cardboard box, with all sides rectangles, has edges of length x, $2\,x$ and y.

(a) Given that all 6 sides of the box are made of cardboard, what is the area of cardboard in terms of x and y?

(b) What is the volume of the box in terms of x and y?

(c) If the volume of the box is fixed as $1m^3$, find an expression for y in terms of x.

(d) A box manufacturer wishes to make a box of the sides described, of volume $1m^3$, using the minimal amount of cardboard. What are the dimensions of the box which minimise the amount of cardboard?

3. In a shooting attraction at a fairground, participants have to shoot the centre of a target which is moving up and down in a straight line.

The gun is held at a fixed height h metres above the ground where $h > 1$. The angle between the gun and the horizontal can be freely varied. The target starts mid way between its highest and lowest positions and if t denotes the time elapsed since the target started moving, its height above the ground in metres is given by $h + \sin(t)$. The horizontal distance between the gun and the target is 10m. The gun fires bullets, of unit mass, with a speed u m/s.

Given that the gun is fired when the target is at its highest possible position, show that if α is the angle of the gun to the horizontal then the bullet will hit the target if α satisfies,

$$-50g - u^2 \cos\left(\frac{10}{u}\sec(\alpha)\right) + 10u^2\tan(\alpha) - 50g\tan^2(\alpha) = 0,$$

where g denotes the acceleration due to gravity.

4. The sides of a right angled triangle obey the equation $a^2 + b^2 = c^2$, where c is the hypotenuse. Show that there are infinitely many such right angled triangles where the sides are all integers, and where $c = a + 2$. Can you give a formula covering all these cases?

5. Suppose that the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are horizontal and that the unit vector $\boldsymbol{k}$ points vertically up.

A particle of mass m moves so that relative to some fixed origin O its velocity at time t is given by $\boldsymbol{v}$ where,

$$\boldsymbol{v} = \left(e^{-kt}\beta\cos\phi\right)\boldsymbol{i} + \left(\frac{1}{k}(g + k\beta\sin\phi)e^{-kt} - \frac{g}{k}\right)\boldsymbol{k}.$$

Here β, ϕ, k are all fixed numbers with $\beta, k > 0$ and g denotes the acceleration due to gravity.

(i) Explain the physical significance of β and ϕ.

(ii) Show that the force acting on the particle is given by,

$$-mg\boldsymbol{k} - mk\boldsymbol{v}.$$

(iii) Find the position of the particle at time t and describe how the particle will behave for very large t.

6. An island is populated entirely by Knights and Knaves, where Knights always tell the truth, and Knaves always lie. On the island you meet a group of three people A, B and C. Then A and B make the following statements:

A says 'At least two of us are Knaves'.

B says 'Both A and C are Knaves'.

a) Is A a Knight or a Knave, or is there insufficient information to tell?

b) Is B a Knight or a Knave, or is there insufficient information to tell?

Sometime later you meet two more inhabitants X and Y.

Y says 'If I am a Knight then X is a Knave'.

c) Is X a Knight or a Knave, or is there insufficient information to tell?

Give reasons for your answers.

7. On the occasion of a royal tournament, there was an entry of $2n$ knights, where n is a positive integer.

a) The first thing they all had to do was to ride past the king in single file. How many possible orders were there for the knights to ride past in?

b) Next the heralds organised the knights into pairs for jousting. Each pair was labelled with the time at which its joust was to take place. In how many possible ways could they have done this pairing? (Note that the order within each pair does not matter.)

c) As each contest was completed, the heralds announced the result to the crowd. Given that we now know who was in each pair, how many possible results could there be from the whole days competition of n jousts (one for each pair). Note that a draw was not an acceptable outcome.

Now n of the knights (who before had all been distinguishable because they had different shields) are given identical armour and shields, so that they can no longer be told apart.

d) In how many ways could the n knights being given identical armour and shields be chosen from the $2n$ knights?

e) Finally the king chooses n of the $2n$ knights to go on a quest (the order of choice does not matter, only whether they are chosen or not). In how many distinguishable ways could the king do this? Remember that n of the knights cannot be told apart. [You may leave this answer as a sum, rather than looking for a single expression.]

8. Four fair coins are thrown one after the other. What is the probability of the following three events?

a) There are exactly 3 heads.

b) There are strictly more heads than tails.

c) The first two coins are heads and the total number of heads is three.

Now four fair coins are thrown one after the other as before, but if the fourth coin is a head, and only in that case, a fifth coin is tossed. For this new game;

d) What is the probability that there are exactly 3 heads?

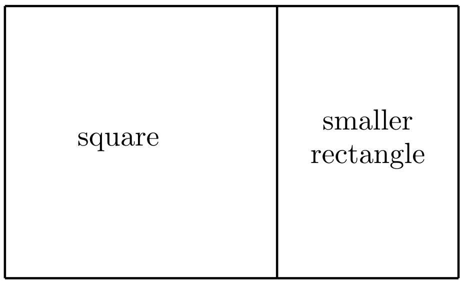
e) What is the probability that the last coin tossed is a tail?

9:

(a) A rectangular sheet of paper has sides of length a and b, with $a > b$. It is cut in half to make two smaller rectangular sheets of paper, each of exactly the same size and shape. The ratio of the lengths of sides of the smaller sheets are the same as the ratio for the original sheet. What is the ratio of the sides? [This is the principle behind the $A3$, $A4$ etc. paper sizes.]

(b) A rectangular cuboid (i.e. all angles are right angles and all faces are rectangles) has lengths of sides c, d, e with $c > d > e$. It is cut in half along the longest side c to make two rectangular cuboids of exactly the same size and shape. The ratios of the sides of the smaller cuboids are the same as the ratios for the original cuboid, i.e. the fractions longest/middle are the same for the cuboids, and the same for middle/shortest. What are the ratios of the sides? [You may assume that the lengths of the sides of the smaller cuboid have order $d > e > c/2$.]

(c) Another rectangular sheet of paper has the property that if it is cut into a square and a smaller rectangle as shown in the picture, then the smaller rectangle has the same ratio of sides as the original rectangle. Find this ratio.



END OF EXAM